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University Examinations 2012/2013

THIRD YEAR, SECOND SEMESTER EXAMINATIONS FOR THE DEGREE OF BACHELOR OF SCIENCE IN MATHEMATICS AND COMPUTER SCIENCE, BACHELOR OF SCIENCE IN STATISTICS AND BACHELOR OF SCIENCE IN ACTUARIAL SCIENCE

STA 2300/SMA 2330: THEORY OF ESTIMATION

DATE: AUGUST 2013 **TIME: 2 HOURS**

INSTRUCTIONS: *Answer question **one** and any other **two** questions*

QUESTION ONE (30 MARKS)

a) Define the properties of estimators. (5 Marks)

b) Let $x_1, \dots x_n$ be iid Poisson (λ) where ($\lambda < 0$) is the unknown parameter.

   i. Find the maximum likelihood estimator for λ. (4 Marks)

   ii. Find UMVUE (uniformly minimum variance unbiased estimator) for λ. (2 Marks)

c) Let $x_1, \dots x_n$ be iid Bernoulli (p) where $0 < p < 1$ is the unknown parameter. Consider T(p)=p

   i. Show that the cramer Rao lower bound of p is the same as var($\bar{x}$) and thus $\bar{x}$ is the UMVUE of p. (4 Marks)

   ii. Consider the specific statistic $T = \sum_{i=1}^{n} x_i$ for $t \in T\{0,1,2, \dots . n\}$. Verify that T is sufficient for P by showing that the conditional distribution of $(x_1, \dots x_n)$ given $T = t$ does not involve p(5).

   iii. With $n \geq 2$ consider an estimator $T = {}^{1}/_{2}\,(x_1 + x_2)$ which is a biased estimator for p. show that through Rao-Blackwelization process, one again ends up with a refined unbiased estimator of P i.e $\bar{x}$. (3 Marks)
   Similarly for n=2 there is no improvement over T. (2 Marks)

   iv. Show that the estimator $T = \sum_{i=1}^{n} x_i$ is minimal sufficient for p by Lehman-scheffe theorem. (3 Marks)

   v. Let $u = x_1 x_2 + x_3$ is u a sufficient statistic for p? (2 Marks)

QUESTION TWO (20 MARKS)

a) Let $x_1, \dots x_n$ be iid random variable with pdf $f(.;\theta)\theta \in \ominus \subseteq \mathbb{R}$ and consider the squared loss function, for an estimator
$T = T(x_1, \dots x_n)$,

$$L(\theta;T) = L[\theta; T(x_1, \dots x_n)\,] = [\theta - T(x_1, \dots x_n)\,]^2$$

Let θ be a random variable with prior pdf $\curlywedge$. Determine T so that it is a Bayes estimate of θ . (8 Marks)

b) Let $x_1, \dots x_n$ be iid random variable from Bernoulli (θ), $\theta \in \ominus = (0,1)$. Choose $\curlywedge$ to be the Beta density with parameter $\propto$ and β, that is

$$\curlywedge(\theta) = \begin{cases} \frac{\Gamma(\propto + \beta)}{\Gamma\propto \Gamma\beta} & \theta^{\propto - 1}(1-\theta)^{\beta - 1} \quad \theta \in (0,1) \\ 0 & otherwise \end{cases}$$

Find the Bayes estimate of θ (8 Marks)

c) Suppose there is a prior pdf $\curlywedge$ on $\ominus$ such that for the Bayes estimate T defined by

$$T(x_1, \dots x_n) = \frac{\int_{\ominus} \theta f(x_1;\theta) \dots . f(x_n;\theta)\curlywedge(\theta)d\theta}{\int_{\ominus} f(x_1;\theta) \dots f(x_n;\theta)\curlywedge(\theta)d\theta}$$

The risk $R(\theta;T)$ is independent of θ. Show that T is minimax. (4 Marks)

QUESTION THREE (20 MARKS)

a) Suppose that $x_1, \dots x_n$ are iid $N(\mu, \sigma^2)$ where μ and σ^2 are both unknown, $\theta = (\mu, \sigma^2) - \infty < \mu < \infty, 0 < \sigma < \infty, n \geq 2$ where $\chi = \mathbb{R}$ and $\ominus = \mathbb{R} \times \mathbb{R}^+$.
Find MLE for θ. (5 Marks)

b) State the invariance property of MLE. (2 Marks)

c) Consider a population described as $N(\mu, \sigma^2)$ where $\mu \in \mathbb{R}$ is unknown but $\sigma \in \mathbb{R}^+$ is known. Consider the following estimators of μ

$$T_1 = x_1 + x_2, \qquad T_2 = \frac{1}{2}(x_1 + x_3) \;\; T_3 = \bar{x}\,, \;\; T_4 = \frac{1}{3}(x_1 + x_3)$$

$$T_5 = x_1 + T_2 - x_4 \qquad T_6 = \frac{1}{10}\sum_{i=1}^{4} ix_i$$

i. Show that T_1 and T_4 are both biased estimators of μ but $T_2, T_3\; T_5 \; and \; T_6$ are unbiased estimators. (4 Marks)

ii. Consider the unbiased estimators, Compute the mean squared error (MSE) for each, and hence determine the best unbiased estimator amongst them. (4 Marks)

iii. Compute the MSE for the biased estimators T_1 and T_4. (2 Marks)

iv. Show that any statistic which is one to one function of a minimal sufficient statistic is itself minimal sufficient. (3 Marks)

QUESTION FOUR (20 MARKS)

a) Let $x_1, \dots\dots. x_n$ be iid random $N(\mu, \sigma^2)$ where $\theta = (\mu, \sigma^2)$ and both μ and σ are unknown. Where $\chi = \mathbb{R}$ and $\theta = \mathbb{R} \times \mathbb{R}^+$. Find the minimal sufficient statistic for θ. (4 Marks)

b) Suppose that T=T(X) is an unbiased estimator of a real valued parametric function
   a. $T(\theta)$ such that its derivative exists $\forall\theta \in \ominus$
   b. Show that
   c. $V_\theta(T) \geq \frac{\{T'(\theta\}^2}{nE_\theta\left[\left\{\frac{d}{d\theta}[\log f(X_1;\theta)]\right\}^2\right]}$ (8 Marks)

c) Let T be an unbiased estimator of a real valued parametric function $T(\theta)$where the unknown parameter $\theta \in \mathbb{R}\underline{C}\ \mathbb{R}^k$.
Suppose that U is a jointly sufficient statistic for θ. Given that the domain space of U is defined as $g(u) = E_\theta[T|U = u]$, for $u \epsilon U$ show that. (2 Marks)
   i. If $w = g(u)$, then W is an unbiased estimator of $T(\theta)$. (2 Marks)
   ii. $V_\theta(w) \leq V_\theta(T)\ \forall\theta \in \ominus$, with the equality holding iff T is the same as W. (3 Marks)

d) Define
   i. Prior distribution (1 Mark)
   ii. Posterior distribution. (2 Marks)