MURANG'A UNIVERSITY COLLEGE

THIRD YEAR FIRST SEMESTER EXAMINATION FOR THE DEGREE OF

BACHELOR OF SCIENCE IN MATHEMATICS AND COMPUTER SCIENCE

SMA 2303 : GROUP THEORY

DATE: 9 TH DECEMBER 2015 **TIME 2 HOURS**

INSTRUCTIONS: ANSWER QUESTION ONE (COMPULSORY) ANY OTHER TWO QUESTIONS

QUESTION ONE (30 MARKS)

a) Let G = S_3 and H = <(12)>. Find H(13) and (13)H (3 Marks)

b) Let G be a group and $\emptyset : G \to H$ be a group homomorphism. Show that

i. If $a \in G$, then $\emptyset(a^{-1}) = [\emptyset(a)]^{-1}$ (3 Marks)

ii. $\emptyset$ is one to one if ker $\emptyset$ = {e} (4 Marks)

c) Define the term odd and even permutation. (3 Marks)

d) Express (1234) as a product of transpositions. (3 Marks)

e) Let G = $\{\pm 1, \pm i\}$. Draw the Cayley table for (G,$\times$) and find the order of all elements. (4 Marks)

f) Show that the centre of a group is a normal subgroup (5 Marks)

g) Prove that in a group, both the identity and inverses are unique. (5 Marks)

QUESTION TWO (20 MARKS)

a)

i. Define the term subgroup. (2 Marks)

ii. Show whether or not {1, 2, 4} is a subgroup of ($\mathbb{Z}_5{}^*$,$\times$) (2 Marks)

iii. Prove that the union of two or more subgroups of a group G is not necessarily a subgroup of G. (5 Marks)

b)

i. Define the term centralizer of an element in a group. (2 Marks)

ii. Let G = S_3 and a = (12). Show that c (a) = {e, a} (4 Marks)

iii. Let $x \in c(a)$. Show that x^{-1} and x^k are $c(a)$ (5 Marks)

QUESTION THREE (20 MARKS)

a)

i. Define the term "order of an element" in a group. (3 Marks)

ii. Let G be a group and suppose an element $a \in G$is of order m. Show that $a^s = e$ if and only if $m|s$. (6 Marks)

b)

i. Define the term cyclic group. Give two examples of cyclic group and state why they are cyclic. (5 Marks)

ii. Let G = {$a \in \mathbb{Z}_{10}$ | gcd (a,10) = 1}. Find all inverses of the elements in G and hence find any generator of G, also show whether 3 is a generator of G. (6 Marks)

QUESTION FOUR (20 MARKS)

a) Define the group D_4 (2 Marks)

b) Let $\alpha, \beta \in S_5$ defined by $\propto = \begin{pmatrix} 12345 \\ 25314 \end{pmatrix}$, $\beta = \begin{pmatrix} 12345 \\ 54123 \end{pmatrix}$ and $\gamma = \begin{pmatrix} 12345 \\ 53241 \end{pmatrix}$

Find $\alpha_o\beta_o\gamma$ and $\alpha_o\beta_o\alpha_o\gamma$ (5 Marks)

c) Let G be a group and $\emptyset : G \to H$ be a group homomorphism with image$\emptyset(G)$. Show that

i. If G is cyclic then $\emptyset(G)$ is cyclic. (4 Marks)

ii. If G is abelian then $\emptyset(G)$ is abelian. (4 Marks)

d)

i. Define the term transposition. (2 Marks)

ii. Express (1325)(1246)(36) as a product of disjoint cycles. (3 Marks)

QUESTION FIVE (20 MARKS)

a) Prove that every cyclic group is abelian. Determine whether the converse holds [4 marks]

b) Let $K = \{p, 2, q\}$ and let $*$ be a binary operation on K. Suppose the Cayley table for $< K,* >$ is as shown below. Determine (giving reasons) whether $< K,* >$ is a group [4 marks]

$*$	p	2	Q
P	2	q	P
2	q	p	2
q	p	2	Q

c) Let S=$\{1, -1, i, -i\}$. Show that S is a group under usual multiplication. Hence show that.

$< S,\times > = < i > = < -i >$ [6 marks]

d) Define the following terms and give an example in each

i. Group orbit of an element (2 Marks)

ii. Stabilizer of an element (2 Marks)

iii. Centre of a group (1 Marks)

iv. Nilpotent element (1 Marks)