MURANGA UNIVERSITY SOLLEGE

THIRD YEAR SEMESTER I EXAMINATION FOR THE DEGREE OF BACHELOR OF SCIENCE IN MATHEMATICS AND COMPUTER SCIENCE

SMA 2301 REAL ANALYSIS I

DATE: 16^{TH} DECEMBER 2015 TIME 2 HOURS

Instructions: Answer Question One (Compulsory) and Any Other Two Questions

QUESTION ONE (30 MARKS)

a) Define the following terms:

i. Set (1 Mark)

ii. Countably infinite set (2 Marks)

b) Define the term Limit Point. Hence show whether the set of positive integers has limit points (4 Marks)

c) Show that the sequence $(X_n) = 1 + (-1)^n(\frac{1}{n^2}),\ n \in \mathbb{N}$ converges to 1 in $\mathbb{R}$ (4 Marks)

d) Define the following terms:

i. Monotonic increasing sequences (2 Marks)

ii. Cauchy Sequence (2 Marks)

e) Let A = $\mathbb{R}$ and d: $A \to A$ be defined as d(x, y) = $|x - y|$ $\forall\, x, y \in A$. Show if d is a metric (4 Marks)

f) Let (X, d) be a metric space. Prove that any subset A of X which has a finite number of elements cannot have a limit point (3 Marks)

g) Every infinite subset F of a compact set K has a limit point in K. Prove. (4 Marks)

h) Two functions f and g are defined as follows:

$$f : \mathbb{N} \longrightarrow \mathbb{N},\ f(n) = 2n$$

$$g : \mathbb{N} \longrightarrow \mathbb{N},\ g(n) = \begin{cases} 2^{n-1}, & if \quad n \quad is\ oddd \\ 3^{n-2}, & if \quad n \quad is\ even \end{cases}$$

Determine which of the functions is one to one. Justify your answer. (4 marks)

QUESTION TWO (20 MARKS)

a)Define the term convergent sequence. Hence show if the sequence

$(X_n) = \begin{cases} 1, & if\ n \quad is\ even \\ 0, & if\ n \quad is\ odd \end{cases}$ converges (4 Marks)

b) Let (X_n) be a sequence of real numbers.

i. Prove that (X_n) converges if and only if $\overline{lim}\ X_n = \underline{lim}\ X_n$ and that this number is real (5 Marks)

ii. Let $(X_n) = \{1 + \frac{1}{2n}\}$, find the limit point of (X_n) (1 Marks)

iii. Let $(X_n) = (\frac{1}{n})$. Find the $\overline{lim}\ X_n$ and $\underline{lim}\ X_n$ (4 Marks)

c)

i. Prove that every convergent sequence is Cauchy (4 Marks)

ii. Give a counter example of a Cauchy sequence which does not converge (2 Marks)

QUESTION THREE (20 MARKS)

a)

i. Define the term metric space. Give an example of a metric space. (3 Marks)

ii. Let (X, d) be a metric space. Show that the whole space X and the empty set ∅ are open (5 Marks)

b)

i. Define the term complement of a set A. (1 Marks)

ii. Let (X, d) be a metric space and $A \subset X$. Show that *A* is closed if A^c is open (6 Marks)

c) Use a counter example or otherwise to show that an infinite intersection of open sets need not be open. (5 Marks)

QUESTION FOUR (20 MARKS)

a)

i. Define the term bounded set. (2 Marks)

ii. Construct a bounded set of real numbers with exactly 3 limit points (5 Marks)

b)

i. Define the term compact set (2 Marks)

ii. Show that (0, 1) is not a compact subset on $\mathbb{R}$ (5 Marks)

c) Let E be a closed and bounded subset of $\mathbb{R}$. Prove that both sup E and Inf E belong to E

(6 Marks)

QUESTION FIVE (20 MARKS)

a) Given reasons, state whether each of the following statements is true or false.

i) The isolated point of $\mathbb{Z}$ are the integers themselves. (2marks)

ii) The boundary points of $\mathbb{Z}$ are all integers. (2marks)

b)

i. Find where possible the supremum and infimum of each of the sets below.

(4marks)

$A = \left\{\frac{1}{2}, -\frac{2}{3}, \frac{3}{4}, -\frac{4}{5}, \frac{5}{6}, -\frac{6}{7}, \dots\right\}$ And $B = \left\{0, \frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \frac{5}{3}, \frac{6}{3}, \dots\right\}$

ii. Prove that if a set has a supremum, then then supremum is unique. (5 marks)

c) Consider the set $A = (1,3) \cup \{5\}$. Determine

i) The interior of A. (3marks)

ii) The closure of A. (4marks)