EMBU UNIVERSITY COLLEGE

(A CONSTITUENT COLLEGE OF THE UNIVERSITY OF NAIROBI)

TRIMESTER EXAMINATIONS 2013/2014

SECOND YEAR EXAMINATION FOR THE DEGREE OF BACHELOR OF SCIENCE

SMA 205: INTRODUCTION TO ALGEBRA

DATE: AUGUST 11, 2014 **TIME: 8.30 – 10.30AM**

INSTRUCTIONS:

Answer Question ONE and ANY Other TWO Questions.

QUESTION ONE

a) Let the binary operation * be defined on $S = \{a, b, c, d, e\}$ by means of the table below

*	a	b	c	d	e
a	a	b	c	b	d
b	b	c	a	e	c
c	c	a	b	b	a
d	b	e	b	e	d
e	d	b	a	d	c

Compute:

i) $(b * d) * c$ and $b * (d * c)$ from the table. Can you say on the basis of this computation whether $*$ is associative (2 marks)

ii) Is $*$commutative? Why? (1 mark)

b) i) Determine whether the definition of $*$ given below does give a binary operation on the given set

- On $\mathbb{Z}^+$, define $*$ by $a * b = c$ where c is at least 5 more than $a + b$. (2 marks)

ii) (I) Define an identity element e for a given at S (1 mark)

(II) Show that 1 is the identity element with respect to the binary operation $*$ given by

$x * y = x + y - 1 \ \forall \, x, y \in \mathbb{Z}$ (2 marks)

c) i) The table in the figure below defines a binary operation $*$ on the set $S = \{A, B, C, D\}$

*	A	B	C	D
A	B	C	A	B
B	C	D	B	A
C	A	B	C	D
D	A	B	D	D

From the table, determine whether S is a group with respect to $*$ (4 marks)

ii) Distinguish a field and a ring as sets of algebraic structures (2 marks)

d) i) State the law of trichotomy for integers (2 marks)

ii) Define the order relation $a < b$ for $a, b \in Z$. Hence prove that if $a, b, c \in Z$ and are such that $a < b$ *and* $0 < c$, then $ac < bc$. (3 marks)

e) (i) When is a relation R on a set A said to be an equivalence relation? (2 marks)

(ii) Show that the relation R defined on the set of integers Z by xRy iff $|x| = |y|$ is an equivalence relation. (3 marks)

f) For the integers $a = -357$ and $b = 13$, find the values of q and r that satisfy the conditions in the Division Algorithm (4 marks)

g) Prove that if $a \equiv b \ (mod\ n)$ and x is any integer, then $ax \equiv bx \ (mod\ n)$ (3 marks)

QUESTION TWO

a) Consider the binary operation defined on $S = N$ by $a * b = a$.

i) Using appropriate examples show that the binary operation $*$ is not commutative but is associative (3 marks)

ii) Show that for the above binary operation, there is no identity element. (4 marks)

b) If $n \geq 2$, let $\mathbb{Z}_n$ denote the set $\{0,1, \dots\dots, n-1\}$ under the operation of multiplication modulo n.

i) Construct the multiplication tables for $\mathbb{Z}_5$ and $\mathbb{Z}_6$ and using the tables determine which set is a field. (8 marks)

ii) Use the table for Z_5 to divide $2\ by\ 3$. (2 marks)

c) Find the inverse of the following matrix, whose entries are elements of $\mathbb{Z}_7$:

$$A = \begin{bmatrix} 5 & 2 \\ 6 & 3 \end{bmatrix}$$ (3 marks)

QUESTION THREE

a) i) State the 5 postulates for positive integers. (5 marks)

ii) Prove that zero (0) is the unique additive identity on $\mathbb{Z}$. (3 marks)

b) i) Find the gcd(a, b) of 1492 and 1776 using the Arithmetic for the Euclidean Algorithm. Hence express the gcd in the form $gcd(a, b) = am + bn,\ m, n \in \mathbb{Z}$. (7 marks)

ii) Show that the integers m and n in equation $gcd(a, b) = am + bn$ are not unique (3 marks)

c) State without proof the Fundamental Theorem of Arithmetic (2 marks)

QUESTION FOUR

a) i) Define the absolute value $|a|$ of an integer a (1 mark)

ii) Show that for $a, b \in \mathbb{Z}$, $|a + b| \leq |a| + |b|$ (4 marks)

b) i) State the well ordering property for integers (2 marks)

ii) Prove that 1 is the least positive integer (4 marks)

iii) Use the principle of mathematical induction to prove that

$$1^3 + 2^3 + 3^3 + \cdots n^3 = \frac{n^2(n+1)^2}{4}$$ (6 marks)

QUESTION FIVE

a) Prove that for the integers a, b *and* c, if a divides both b and c, the it must also divide $ub + vc, \forall\, u, v \in \mathbb{Z}$. (2 marks)

b) Solve the following system of linear congruencies

$$3x + 7y \equiv 4 \ (mod\ 11)$$
$$8x + 6y \equiv 1 \ (mod\ 11)$$

(7 marks)

c) Show that the relation "congruence modulo 4" defined on the set $\mathbb{Z}$ of all integers is an equivalence relation. (5 marks)

d) Find all solutions to the linear congruence $20x \equiv 14 \ (mod\ 63)$ (6 marks)

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