EMBU UNIVERSITY COLLEGE

(A Constituent College of the University of Nairobi)

2015/2016 ACADEMIC YEAR

FIRST SEMESTER EXAMINATION

THIRD YEAR EXAMINATION FOR THE DEGREE OF BACHELOR OF SCIENCE

SMA 303: ALGEBRA I

DATE: DECEMBER 9, 2015 **TIME: 14:00-16:00**

INSTRUCTIONS:

Answer Question ONE and ANY Other TWO Questions.

QUESTION ONE:

a) Show that if H and K are subgroups of a group G, then $H \cap K$ is a subgroup of G.

(4 marks)

b) Given a normal subgroup N of a group G and let $x, y \in G$. Show that $(xN)(yN) = (xy)N$

(4 marks)

c) Show that every cyclic group of order $n \in \mathbb{N}$ is isomorphic to $(\mathbf{Z}_n, +)$

(4 marks)

d) Given a group $G = \langle a \rangle$ and $|a| = 15$. Find all the generators and distinct subgroups of the group G (4 marks)

e) Show that if $n \geq 3$ then S_n is non abelian (4 marks)

f) If U, V are ideals of a ring R, let $U + V = \{u + v \mid u \in U, v \in V\}$. Show that $U + V$ is also an ideal of R. (4 marks)

g) Show that any field is an integral domain (4 marks)

h) Define the following terms

i) Kernel of a ring homomorphism (1 mark)

ii) A prime ideal (1 mark)

QUESTION TWO

a) Define the center of a group G and show that it is a subgroup. (7 marks)

b) Define an ideal I of a ring R. Show that for a commutative ring R with unity element e, the set $(a) = \{ar \mid r \in R\}$ is an ideal of R (8 marks)

c) Prove that in any group, the orders of ab and ba are the same (5 marks)

QUESTION THREE

a) Let (G, ∗), (H, †) and (K, ‡) be groups, and $\alpha : G \to H$, $\beta : H \to K$ be homomorphisms. Show that:

i. $\beta \circ \alpha : G \to K$ is also a homomorphism. (4 marks)

ii. If α is an isomorphism, then so is $\alpha^{-1} : H \to G$ (5 marks)

iii. If α and β are both isomorphism, then so is $\beta \circ \alpha : G \to K$

(5 marks)

b) State and prove the Lagrange theorem.

(6 marks)

QUESTION FOUR

a) Let $n \in \mathrm{N}$ and consider the group (S_n, $\circ$)

i) Define an even permutation $\alpha \in S_n$ and show that if α and β are two even permutations in S_n, then $\alpha \circ \beta$ is also even. (6 marks)

ii) Consider a permutation $\gamma = (1284)(432)(57)(1423)$ in S_8. Is γ an even permutation? (3 marks)

iii) Write γ as a product of disjoint cycles and determine the order of the subgroup $H = \langle \gamma \rangle$ of S_8 (6 marks)

b) Show that if G is any group, then G is isomorphic to $\mathbf{Z}_{29}$ if and only if $|G| = 29$

(5 marks)

QUESTION FIVE

a) Let $G = \left\{ \begin{bmatrix} \hat{a} & \hat{b} \\ \hat{c} & \hat{d} \end{bmatrix}; \hat{a}, \hat{b}, \hat{c}, \hat{d} \in \mathbf{Z}_5 : \det \begin{bmatrix} \hat{a} & \hat{b} \\ \hat{c} & \hat{d} \end{bmatrix} \neq \hat{0} \right\}$ and $H = \left\{ \begin{bmatrix} \hat{a} & \hat{b} \\ \hat{c} & \hat{d} \end{bmatrix} \in G : \det \begin{bmatrix} \hat{a} & \hat{b} \\ \hat{c} & \hat{d} \end{bmatrix} = \hat{1} \right\}$

i) Assume that G is a group under matrix multiplication. Prove that G is non-abelian and contains 480 elements. (3 marks)

ii) By Cauchy's theorem, G has an element of order 5. Find one.

(3 marks)

iii) Prove that H is a normal subgroup of G. (3 marks)

iv) Prove that for each $m \in G$, $Hm = \{g \in G : \det(g) = \det(m)\}$

(5 marks)

b) State and prove Cayley's theorem on groups of finite order. (6 marks)

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