EMBU UNIVERSITY COLLEGE

(A Constituent College of the University of Nairobi)

2015/2016 ACADEMIC YEAR

FIRST SEMESTER EXAMINATION

THIRD YEAR EXAMINATION FOR THE DEGREE OF BACHELORS OF SCIENCE

SMA 301: REAL ANALYSIS I

DATE: DECEMBER 7, 2015 **TIME: 11:00-13:00**

INSTRUCTIONS:

Answer Question ONE and ANY Other TWO Questions.

QUESTION ONE:

a) Define a neighborhood of a point x in a metric space M. Hence show that if N_1 and N_2 are neighborhoods of x, then their intersection is also a neighborhood of the point x.

(3 marks)

b) (i)When do we say that a sequence x_n is convergent to the limit x in a metric space$M$?

(1 mark)

(ii) Hence prove that if the limit of a sequence exists, then that limit is unique (5 marks)

c) (i) Let X be a non-empty set. What is meant by a metric on X. (3 marks)

(ii) Let X be an arbitrary non-empty set. Show that the function defined by

$d(x,y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$ is a metric on X (5 marks)

d) Let the metric space $M = R$ and a subset S of M be $S = \boldsymbol{Q}$

Determine:

(i) The closure $\bar{S}$, of S (1 mark)

(ii) The interior $int(S)$, of S (1 mark)

(iii)The boundary δS, of S (1 mark)

e) (i) Distinguish a continuous function and a uniformly continuous function of metric spaces M and N (4 marks)

(ii) Show that composite of continuous functions in metric spaces is continuous.

(Use f^{-1} definition) (3 marks)

f) (i) Define a compact subset A on a metric space (X, d) (1 mark)

(ii) Show that any open interval (a, b) is not compact (2 marks)

QUESTION TWO

a) (i) Define an open set G in a metric space$M$. Hence show that the set $(0,2)$ is open in M.

(4 marks)

(ii) Prove that Intersection of finitely many open sets in a metric space is open. (7 marks)

b) Prove that the complement of an open set is closed and the complement of a closed set is open

(9 marks)

QUESTION THREE

a) (i) Define a Cauchy sequence a metric space M. (1 mark)

(ii) Prove that every convergent sequence is Cauchy, and by use of an appropriate example illustrate that the converse of this is not always true. (6 marks)

b) Show that the set of real numbers R is complete with the usual metric. (13 marks)

QUESTION FOUR

a) State without proof the Cauchy- Schwartz inequality. Hence prove that $\forall\ x, y, z \in \boldsymbol{R}^n$, with a function $d: \boldsymbol{R}^n \times \boldsymbol{R}^n \rightarrow \boldsymbol{R}$ defined by $d(x, y) = \left(\left(\sum_{i=1}^{n}(x_i - y_i)^2\right)\right)^{\frac{1}{2}}$,

$d(x, y) \leq d(x, y) + d(z, y)$. (6 marks)

b) Consider the set $C[0,1]$ which represents real valued continuous functions on the interval $[0,1]$. Define a function $d: C[0,1] \times C[0,1] \rightarrow R$ by $d(f, g) = \int_0^1 |f(x) - g(x)| dx$.

Show that $(C[0,1], d)$ is a metric space. (5 marks)

c) Prove that in any metric space (X, d),

(i) An open sphere is a neighborhood of each and every point in it (5 marks)

(ii) Arbitrary intersection of closed sets is closed (4 marks)

QUESTION FIVE

a) (i) State without proof the Cantor's theorem. (1 mark)

(ii) Show that a function which is uniformly continuous on metric space is continuous on that metric space (3 marks)

b) Show that the function $f(x) = \frac{1}{x}$ is continous on the interval $(0,1)$ but not uniformly continuous on the same interval (5 marks)

c) Let $g: X \to Y and\ f: Y \to Z$ g be two real valued functions such that

   i) g is continuous at $a \in X$

   ii) $g(a) = b \in Y$

   iii) f is continuous $b \in Z$.

Prove that their composite function $(f\ 0\ g)(x) = f(g(x))$ is continuous (Use ε, d definition) (6 marks)

d) Prove that every closed subset of a compact metric space is compact (5 marks)

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