MASENO UNIVERSITY

UNIVERSITY EXAMINATIONS 2016/2017

FIRST YEAR FIRST SEMESTER EXAMINATIONS FOR THE DEGREE OF BACHELOR OF SCIENCE WITH INFORMATION TECHNOLOGY

MAIN CAMPUS

MMA 109: FOUNDATIONS OF PURE MATHEMATICS

Date: 2nd December, 2016 Time: 12.00 – 3.00 pm

INSTRUCTIONS:

- Answer question ONE (30 marks) and any other TWO (20 marks each) questions.
- Proofs should be written carefully.
- Observe further instructions on the answer booklet.
- Cheating is NOT allowed and will be harshly punished.

QUESTION ONE (Compulsory) [30 Marks]

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(a) (i) Define a null set $\emptyset$ and give an example. [2 mks]

(ii) Is the following statement true or false: $\{\emptyset\} = \emptyset$? Why? [2 mks]

(iii) List the elements of the set $A = \{x \in \mathbb{N} : x^2 = 1\}$. [1 mk]

(iv) Let $|A|$ denotes the cardinality of the set A. If $|A| = |B|$, does it imply $A = B$? Explain. [2 mks]

(b) Explain the difference between these two statements, identifying which one is true, if we assume the universe to be $\mathbb{R}$: [2 mks]

First statement: $\forall x, \exists y$ s.t. $x^4 - y = 0$.
Second statement: $\exists y$ s.t. $\forall x, x^4 - y = 0$.

(c) By the aid of truth tables, distinguish between the following statement forms:

(i) Conjunction and Disjunction. [2 mks]

(ii) Implication and Equivalence. [2 mks]

(d) (i) Differentiate between a Lemma and a Corollary. [2 mks]

(ii) Give a counterexample to the statement, "Every odd number is prime". [1 mk]

(e) Consider the statement, "There are infinitely many prime numbers"

(i) Rephrase the statement to be in the form "If $\cdots$, then $\cdots$" [1 mk]

(ii) Give the assumption(s) and the conclusion(s) of the above statement. [2 mks]

(iii) In mathematics, a strong statement is one with weak assumptions and strong conclusions. Make the whole statement in (i) above stronger. [1 mk]

(iv) Prove the statement. [3 mks]

(f) A picture is worth a thousand words. Give a picture proof to the identity: $(a+b)^2 = a^2 + b^2 + 2ab$, where a and b are positive integers. [3 mks]

(g) In mathematics how to write is as important as what to write. Here's a student's proof of the following theorem:

Theorem 1. *If x is an odd integer then x^2 is odd.*

Proof.

$$\begin{aligned} x^2 &= (2n+1)^2 \\ &= 4n^2 - 4n + 1 \\ &= 2(2n^2 - 2n) + 1 \\ &= 2m + 1 \end{aligned}$$

□

Criticize the student's proof and rewrite it. [4 mks]

QUESTION TWO [20 Marks]

(a) Consider the statement, "If n is an integer and $3n+2$ is odd, then n is odd".

(i) State its contrapositive. [2 mks]

(ii) State its converse. [2 mks]

(iii) Prove the statement for all $n \in \mathbb{Z}$. [4 mks]

(b) By the use of quantifiers, connectives and logically equivalent statements, negate the statement: "There is a house on every street such that if that house is blue, the one next to it is black." [4 mks]

(c) Use a truth table to determine whether the statements $P \Rightarrow Q$ and $P \Rightarrow (Q \vee \neg P)$ are logically equivalent or not. [5 mks]

(d) With the aid of an example, explain the meaning of 'Paradox'. [3 mks]

QUESTION THREE [20 Marks]

(a) Fermat's last conjecture was turned into Fermat's last theorem by Andrew Wiles in 1995.

(i) Distinguish between a conjecture and a theorem. [2 mks]

(ii) State the Fermat's Last Theorem. [2 mks]

(b) By giving a counterexample, disprove Fermat's conjecture that numbers of the form $2^{2^m}+1$, where m is a nonnegative integer are all primes. [3 mks]

(c) Prove that if n^2 is divisible by 3, then n is divisible by 3. Hence or otherwise, show that $\sqrt{3}$ is irrational. [8 mks]

(d) Define the absolute value of a real number, and hence prove that for all $x, y \in \mathbb{R}$, $||x| - |y|| \leq |x-y|$. [5 mks]

QUESTION FOUR [20 Marks]

(a) (i) State precisely the Principle of Mathematical induction. [3 mks]

(ii) Using the principle stated in (i) above, prove the inequality

$$3^n < n!$$

for all positive integers n such that $n > 6$. [8 mks]

(b) Give a direct proof to the statement: The product of two real negative numbers is positive. [3 mks]

(c) Prove that if n is an integer, then n^2+3n+2 is an even integer. [6 mks]

QUESTION FIVE [20 Marks]

(a) (i) Explain the concept of proof by Contrapositive. [3 mks]

(ii) Using the technique of proof by contrapositive, prove that if x^2 is an even integer then x is an even integer too. [7 mks]

(b) Prove that for all natural numbers n, the expression $n^3 + 3n + 2n$ is divisible by 6. [4 mks]

(c) By using an appropriate proof technique, show that [6 mks]

$$\sum_{i=1}^{n} (3^i - 3i + 1) = n^3.$$

END

ALL THE BEST