MASENO UNIVERSITY

UNIVERSITY EXAMINATIONS 2016/2017

SECOND YEAR FIRST SEMESTER EXAMINATIONS FOR THE DEGREE OF BACHELOR OF SCIENCE WITH INFORMATION TECHNOLOGY

MAIN CAMPUS

MMA 219: GRAPH THEORY

Date: 10th December, 2016 Time: 12.00 - 3.00 pm

INSTRUCTIONS:

- Answer question ONE and any other TWO questions.

QUESTION ONE (Compulsory) [30 Marks]

(a) Show that if two graphs G and G' are isomorphic, then the order of G is equal to the order of G' and the size of G is equal to the size of G'. [4 Marks]

(b) Find the complements of K_n and $K_{m,n}$. [3 Marks]

(c) Prove that the sum of the degrees of a graph is twice the number of edges in it. Hence, show that every graph has an even number of odd vertices. [7 Marks]

(d) Prove that the number of edges, m, of a simple graph on n vertices with k components satisfies the following inequality: $m \geq (n-k)$. [4 Marks]

(e) Differentiate between:

(i) Eulerian graph and Hamiltonian graph. [2 Marks]

(ii) girth and clique of a graph. [2 Marks]

(f) Use induction to show that if T is a tree on n vertices then it must have $n-1$ edges. Hence prove that if G is a forest with n vertices and k components, then G has $n-k$ edges. [4 Marks]

(g) Show that the complete graph K_5 is non-planar. [4 Marks]

QUESTION TWO [20 Marks]

(a) State Cayley' theorem and hence show that show that the number of trees on n vertices in which a given vertex is of degree k, where $0 < k < n$, is given by $\binom{n-2}{k-1}(n-1)^{n-k-1}$. [6 Marks]

(b) Find the number of trees on n vertices with distinct labels such that a given vertex is a leaf. [4 Marks]

(c) Deduce that, if n is large, then the probability that a given vertex of a tree with n vertices is a leaf is approximately e^{-1}. [4 Marks]

(d) Let G be a plane drawing of a connected planar graph, and let n, m and f denote respectively number of vertices, edges and faces of G. Prove that $n - m + f = 2$.

[6 Marks]

QUESTION THREE [20 Marks]

(a) State the following: [6 Marks]

(i) A complete graph which is also a wheel graph.

(ii) A complete graph which is also a cycle graph.

(iii) A complete graph which is also a tree.

(iv) A complete graph which is also a bipartite graph.

(v) A wheel graph that is also a regular graph.

(vi) A null graph which is also complete graph.

(b) Prove that a connected graph G is Eulerian if and only if the degree of each vertex of G is even. [8 Marks]

(c) Can someone cross all the bridges shown in the map below exactly once and return to the starting point? Support your answer with an explanation. [3 Marks]

(d) Show that any simple graph on n vertices with more than $(n-1)(n-2)/2$ edges is connected. [3 Marks]

QUESTION FOUR [20 Marks]

(a) Use combinatorial arguments to find the total number of labelled graphs on $[n] := \{1, 2, \ldots, n\}$ and also the number of these graphs with exactly k edges.

[4 Marks]

(b) Using techniques from graph theory, show that $1 + 2 + \cdots + n = n(n+1)/2$.

[6 Marks]

(c) Show that, in any gathering of six people, there are either three people who all know each other or three people none of whom knows either of the other two.

[7 Marks]

(d) Show that, for each value of n, the graph associated with the alcohol $C_nH_{2n+1}OH$ is a tree (the oxygen vertex has degree 2). [3 Marks]

QUESTION FIVE [20 Marks]

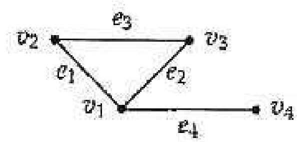
(a) State Kuratowski's theorem. [2 Marks]

(b) Show that if a simple graph G has at least 11 vertices, then both the graph G and its complement cannot be planar graphs. [3 Marks]

(c) Show that a planar graph is bipartite if and only if its dual is Eulerian. [7 Marks]

(d) (i) State matrix-tree theorem. [1 Mark]

(ii) Find the adjacency matrix, incidence matrix and Laplacian matrix of the following graph.



Hence determine the number of spanning trees of the graph. [7 Marks]