MASENO UNIVERSITY

UNIVERSITY EXAMINATIONS 2015/2016

FIRST YEAR FIRST SEMESTER EXAMINATIONS FOR THE DEGREE OF BACHELOR OF SCIENCE WITH INFORMATION TECHNOLOGY

MAIN CAMPUS

MMA 109: FOUNDATIONS OF PURE MATHEMATICS

Date: 12th January, 2016 Time: 11.00 - 1.00 pm

INSTRUCTIONS:

- Answer question ONE and any other TWO questions.
- Observe further instructions from the booklet.

QUESTION ONE (Compulsory) [30 Marks]

(a) In mathematics how to write is as important as what to write. Here's a student's proof of the following theorem:

Theorem 1. *If x is an even integer then x^2 is even.*

Proof.

$$\begin{aligned} x^2 &= (2n)^2 \\ &= 4n^2 \\ &= 2(2n^2) \\ &= 2m \end{aligned}$$

□

Criticize the student's proof and rewrite it. [4 Marks]

(b) Explain the difference between these two statements, identifying which one is true, if we assume the universe to be $\mathbb{R}$.

First Statement: $\forall x, \exists y$ such that $x^2 + y = 0$.

Second Statement: $\exists y$ such that $\forall x,\ x^2 + y = 0$. [3 Marks]

(c) Consider the statement, "If y is even then $5x + y$ is even or x is odd."

(i) State the contrapositive. [1 Mark]

(ii) State the converse. [1 Mark]

(iii) Prove the statement for all $x, y \in \mathbb{Z}$. [4 Marks]

(d) (i) Differentiate between a lemma and a corollary. [2 Marks]

(ii) Give a counterexample to the statement, "Every prime number is odd". [1 Mark]

(iii) A picture is worth a thousand words. Give a 'picture proof' to the statement:

$$(1 + 2 + \cdots + n)^2 = 1^3 + 2^3 + \cdots + n^3$$

where $n \in \mathbb{N}$. [3 Marks]

(e) Use the principle of mathematical induction to show that $n! \geq 2^{n-1}$ for $n \in \mathbb{N}$. [4 Marks]

(f) Define $f : \mathbb{R} \longrightarrow \mathbb{R}$ by $f(x) = 8x - 5$. Show that f is a bijection. [4 Marks]

(g) Let $S = \{1, 2, 3\}$ and define an equivalence relation R on S by

$$xRy \text{ if and only if } x + y \text{ is odd.}$$

Find the equivalence class of each element of S. [3 Marks]

QUESTION TWO [20 Marks]

(a) Prove that $\{x \in \mathbb{R} : x^2 - 1 = 0\} = \{1, -1\}$. [6 Marks]

(b) Define a counterexample. Hence, by giving a counterexample, disprove Fermat's conjecture that numbers of the form $2^{2^m} + 1$, where m is a nonnegative integer are all primes. [3 Marks]

(c) Show that $\sqrt{2}$ is irrational. [5 Marks]

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(c) Show that $\sqrt{2}$ is irrational. [5 Marks]

(d) By the use of quantifiers, connectives and logically equivalent statements, negate the statement, "People who live in glass houses should not throw stones."

[6 Marks]

QUESTION THREE [20 Marks]

(a) Distinguish between a tautology and a contradiction. [2 Marks]

(b) Let x and y be real numbers. Prove that $|xy| = |x||y|$. [7 Marks]

(c) By the use of an example, explain the meaning of: Paradox. [3 Marks]

(d) Use the principle of mathematical induction to show that

$$4\sum_{i=0}^{n} i3^i = (2n-1)3^{n+1} + 3.$$

[8 Marks]

QUESTION FOUR [20 Marks]

(a) By the aid of truth tables, explain the concept of proof by contrapositive.

[4 Marks]

(b) Give a direct proof that if a, b and c are integers such that a divides b and a divides c, then a divides $b + c$. [4 Marks]

(c) Prove that for all natural numbers n, the expression $9^n - 8n - 1$ is a divisible by 64. [8 Marks]

(d) Suppose $f : X \longrightarrow Y$ is a function, and A and B are subsets of X. Prove that $f(A \cap B) \subseteq f(A) \cap f(B)$. [4 Marks]