UNIVERSITY OF EMBU

2017/2018 ACADEMIC YEAR

SECOND SEMESTER EXAMINATIONS

SECOND YEAR EXAMINATION FOR THE DEGREE OF BACHELORS OF SCIENCE, BACHELOR OF EDUCATION (SCIENCE) AND BACHELOR OF SCIENCE (STATISTICS)

SMA 206: INTRODUCTION TO REAL ANALYSIS

DATE: APRIL 12, 2018 **TIME: 8:30 AM – 10:3-0 AM**

INSTRUCTIONS:

Answer Question ONE and ANY other two Questions

QUESTION ONE (30 MARKS)

a) Define an even integer number m. Hence prove that if m^2 is even then m is also even for any integer number m. (4 marks)

b)

i) Prove that if $x, y \in \mathbb{R}$ such that $x < y$ then there is $r \in \mathbb{R}$ such that $x < r < y$ (2 marks)

ii) Prove that for any positive real number x, there exist always a non-negative real number x^{-1}. (3 marks)

c) Define a countable set. Hence illustrate that the set of rational numbers between $[0, 1]$ is countable. (3 marks)

d) Define a Cauchy sequence (x_n) in R. Hence prove that if a sequence (x_n) is convergent then it is Cauchy (4 marks)

e) Determine the accumulation points of each of the set of real numbers

i) The set of natural numbers N;

ii) (a,b]

iii) The set of irrational points (3 marks)

f) Let S be a non-empty subset of R. Prove that the real number A is the sup A iff both the following conditions are satisfied

i) $x \leq A \quad \forall x \subset S$

ii) $\forall \varepsilon > 0 \ \exists x' \in S : A - \varepsilon < x' \leq A$ (3 marks)

g) Discuss the following concepts as used in analysis

i) A partion of a closed interval $[a, b]$ (1 mark)

ii) The Riemanns' upper sum and lower sum of the function f defined on the interval $[a, b]$ (3 marks)

h) State the conditions to be satisfied for a function to be continuous at a point $x = c$. Hence determine whether the function

$$f(x) = \begin{cases} x & 0 \leq x < 1 \\ \frac{1}{2}x & 1 \leq x < 2 \end{cases}$$ is continuous at $x = 1$. (4 marks)

QUESTION TWO (20 MARKS)

a) Given that $x, y \in \mathbb{R}$, show that

i) If x is positive then $-x$ is negative and conversely (4 marks)

ii) $x < y \Rightarrow \frac{1}{y} < \frac{1}{x}$ (3 marks)

iii) $x < y$ iff $x^2 < y^2$ (4 marks)

b)

i. Given that $A \subseteq \mathbb{R}$, show that A is open iff $A = A^0$ (4 marks)

ii. Let $\bar{A}$ denote the closure of a subset $A \subset \mathbb{R}$. Prove that $A = \bar{A}$ if and only if A is closed. (5 marks)

QUESTION THREE (20 MARKS)

a) Prove that if the limit of a function $f(x)$ exists, then that limit is unique (5 marks)

b) Prove that the set of real numbers $\mathbb{R}$ is not countable (5 marks)

c) Using the first principle definition of the limit of a function, prove that

i) $\lim_{n\to\infty}\left(\frac{2n}{2n+2}\right) = 1$ (4 marks)

ii) $\lim_{x\to 3}(x^3 + 2x) = 33$ (6 marks)

QUESTION FOUR (20 MARKS)

a) Prove that every convergent sequence is bounded. Using an appropriate counter example show that the converse of this is not necessarily true (8 marks)

b) Using the sequence $X_n = \left(1+\frac{1}{n}\right)^n): n \in \mathbf{N}$, prove that this sequence is convergent without necessarily determining its limit of convergence. (6 marks)

c) Find the limit superior and limit inferior of the sequence

$X_n = cos\frac{n\pi}{2} + \frac{1}{n}\left(sin\frac{2n+1}{2}\right)\pi : n \in \mathbf{N}$ (6 marks)

QUESTION FIVE (20 MARKS)

a) Using the function $f(x) = |x-4|$ at the point $x = 4$, show that the property of continuity does not necessarily imply differentiability. (7 marks)

b)

i. State the fundamental theorem of definite integral calculus. (2 marks)

ii. By partitioning method, show that the Riemann Integral of the function $f(x) = x$ on the interval $0 \leq x \leq 1$ is $\frac{1}{2}$ (11 marks)

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