UNIVERSITY OF EMBU

2017/2018 ACADEMIC YEAR

SECOND SEMESTER EXAMINATIONS

FIRST YEAR EXAMINATION FOR THE DEGREE OF MASTER OF SCIENCE IN APPLIED MATHEMATICS

SMA 536: APPLIED FUNCTIONAL ANALYSIS

DATE: APRIL 6, 2018 **TIME: 2:00 PM – 5:00 PM**

INSTRUCTIONS:

Answer Question ONE and ANY other two Questions. Exam Duration is 3 Hours.

QUESTION ONE (30 MARKS)

a) Suppose f and g are functions in a space $X = \{f: [0,1] \to \mathbb{R}\}$.
Does $d(f, g) = \max|f - g|$ define a metric? (3 marks)

b) Find all possible topologies on $X = \{a, b\}$. (2 marks)

c) Let X be an inner product space. Simplify $\|x + y\|^2 + \|x - y\|^2$ for all $x, y \in X$. (4 marks)

d) Let X be the closed unit interval $[0,1]$. Then $[0, 1/10)$; $\left(\frac{1}{3}, 1\right)$; and $\left(\frac{1}{n+2}, \frac{1}{n}\right)$, where $n \in \mathbb{Z}$ and $n \geq 2$, is an infinite open cover of $[0,1]$. With clear illustration, find the least finite subcover of X. (4 marks)

e) Show that if X is a normed linear space, then the norm $\|\cdot\|$ defined on X is uniformly continuous. (3 marks)

f) Consider $A = \begin{pmatrix} i & 2 \\ 3 & 4 \end{pmatrix}$ and $T: \mathbb{C}^2 \to \mathbb{C}^2$ be defined as $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} i & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$. Find the adjoint operator of T. (5 marks)

g) Let $T: H \to H$ be a bounded linear operator on a Hilbert space H. Show that

   i) If T is self-adjoint, $\langle Tx, x\rangle$ is real for all $x \in H$. (2 marks)

ii) If H is complex and $\langle Tx, x\rangle$ is real for all $x \in H$, the operator T is self-adjoint. (3 marks)

h) If each of sets A and B is a subset of a linear topological space X, then

$$\text{Int}(A + B) = \text{Int}(A) + \text{Int}(B).$$

Justify this statement. Note: Int( ) denotes interior of the set under brackets. (4 marks)

QUESTION TWO (20 MARKS)

a) Given $f: X \to Y$ and $g: Y \to Z$ to be homeomorphisms, is the composition $g \circ f: X \to Z$ a homeomorphism? explain. (4 marks)

b) A linear topological space is regular. Justify this statement. (6 marks)

c) Show that a normed linear space X together with the topology induced by the norm metric, is a linear topological space. (6 marks)

d) Consider two topological spaces $X = (-1,1)$ and $Y = \mathbb{R}$. Define a continuous map $f: (-1,1) \to \mathbb{R}$ by

$$f(x) = \tan\left(\frac{\pi x}{2}\right).$$

Is f a homeomorphism? Explain. (4 marks)

QUESTION THREE (20 MARKS)

a) Suppose that F is a closed and convex subset of a Hilbert space $\mathcal{H}$. Show that if $x \in \mathcal{H}$, then there is a unique element $P(x)$ in F satisfying (8 marks)

$$\|x - P(x)\| = inf\{\|x - z\|: z \in F\}.$$

The mapping P is called the projection of $\mathcal{H}$ on F.

b) Prove that a bounded linear $P: \mathcal{H} \to \mathcal{H}$ on a Hilbert space $\mathcal{H}$ is a projection if and only if P is self-adjoint and idempotent (that is $P^2 = P$). (8 marks)

c) For all x and y in a normal linear space X, $\big|\|x\| - \|y\|\big| \leq \|x - y\|$. (4 marks)

QUESTION FOUR (20 MARKS)

a) Define a mapping $R: C[0,1] \to C$,

$$Rf = \int_0^1 x\, f(x)dx.$$

i) Show that R is a linear operator. (4 marks)

ii) Consider $C[0,1]$ with the norm,

$$\|f\|_1 = \int_0^1 |f(t)| \, dt.$$

Show that R is bounded. (3 marks)

iii) Show that $\|R\| = 1$. (4 marks)

b) Let X denote a compact Hausdorff space and let $C(X, \mathbb{R})$ denote the set of all continuous functions $f: X \to \mathbb{R}$. Suppose that $\mathcal{A}$ is a closed lattice in $C(X, \mathbb{R})$, and $f \in C(X, \mathbb{R})$, then show that if for every $x, y \in X$, there exists $g_{xy} \in \mathcal{A}$ such that $g_{xy}(x) = f(x)$ and $g_{xy}(y) = f(y)$; then $f \in \mathcal{A}$. (9 marks)

QUESTION FIVE (20 MARKS)

a) Show that the spectrum $\sigma(T)$ of a bounded self-adjoint linear operator $T: \mathcal{H} \to \mathcal{H}$ on a complex Hilbert space $\mathcal{H}$ is real. (8 marks)

b) Let $T: \mathcal{H} \to \mathcal{H}$ be a bounded self-adjoint linear operator on a complex Hilbert space $\mathcal{H}$.

Show that

i) All eigenvalues of T (if they exist), are real. (4 marks)

ii) Eigenvectors corresponding to different eigenvalues of T are orthogonal. (3 marks)

c) Let X denote a metric space. For $x, y \in X$, let $d(x, y)$ denote the distance between x and y. Given $d(x, y) = 0$ if $x = y$ and $d(x, y) = 1$ otherwise, does $d(x, y)$ define a metric? (5 marks)

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