# UNIVERSITY OF EMBU

## 2017/2018 ACADEMIC YEAR

## SECOND SEMESTER EXAMINATIONS

## FIRST YEAR EXAMINATION FOR THE DEGREE OF MASTER OF SCIENCE IN APPLIED MATHEMATICS

## SMA 538: APPLIED DYNAMICAL SYSTEMS I

**DATE: APRIL 10, 2018** **TIME: 2:00 PM – 5:00 PM**

**INSTRUCTIONS:**

**Answer Question ONE and ANY other two Questions. Exam Duration is 3 Hours.**

## QUESTION ONE (30 MARKS)

a) Consider the following vector field:

$$\dot{x} = y,$$
$$\dot{y} = -x + \varepsilon x^2 y.$$

Given that $(x, y) = (0,0)$ is a non-hyperbolic fixed point of the system, determine if this fixed point is stable. (5 marks)

b) Consider the vector field:

$$\dot{x} = y,$$
$$\dot{y} = x - x^3 - \delta y, \qquad \delta \geq 0.$$

Find the range of $\delta$ for the above system to have no closed orbits. (3 marks)

c) Use center manifold theorem to determine the qualitative behavior of the origin for the following system. (5 marks)

$$\dot{x} = xy,$$
$$\dot{y} = -y - x^2.$$

d) Show that the vector field;

$$\dot{x} = Jx + \mathcal{F}(x, \mu), \qquad x \in \mathbb{R}^n, \qquad \mu \in \mathbb{R}^n;$$

can be transformed to a normal form in which $\mathcal{F}(x, \mu)$ satisfies

$$e^{-J^* t}\mathcal{F}\left(e^{J^* t}x, \mu\right) = \mathcal{F}(x, \mu),$$

and

$$\mathcal{F}(0,\mu) \in Ker\, J^*,$$
$$J^* D_x \mathcal{F}(0,\mu) - D_x \mathcal{F}(0,\mu) J^* = 0.$$

(6 marks)

e) Suppose all the eigenvalues of the Jacobian $Df(\bar{x})$ have negative real parts. Use Lyapunov theory to show that the equilibrium solution $x = \bar{x}$ of the nonlinear vector field;

$\dot{x} = f(x),\ x \in \mathbb{R}^n$; is asymptotically stable. (6 marks)

f) If on a simply connected region $D \subset \mathbb{R}^2$, the expression $\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}$ is not identically zero, and does not change sign. Show that

$$\dot{x} = f(x,y)$$
$$\dot{y} = g(x,y),$$

Has no closed orbits lying entirely in $D$. (5 marks)

**QUESTION TWO (20 MARKS)**

Compute the normal form for a vector field on $\mathbb{R}^2$ in the neighborhood of a fixed point where the linear part is given by $J = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. (20 marks)

**QUESTION THREE (20 MARKS)**

a) Use the center manifold theorem to determine the qualitative behavior of the origin, for the system; (8 marks)

$$\dot{x}_1 = -x_2 + x_1 y$$
$$\dot{x}_2 = x_1 + x_2 y$$
$$\dot{y} = -y - x_1^2 - x_2^2 + y^2.$$

b) Consider the map:

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 0 & 1 \\ -\frac{1}{2} & \frac{3}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ -y^3 \end{pmatrix}.$$

The origin is a fixed point of the map.

i) Compute eigenvalues of the map linearized about the origin. (4 marks)

ii) Compute the center manifold. (8 marks)

**QUESTION FOUR (20 MARKS)**

Consider the following vector field;

$$\dot{x} = f(x), \qquad x \in \mathbb{R}^n.$$

Let $\bar{x}$ be a fixed point of the above vector field, and let $V: U \to \mathbb{R}$ be a $C^1$ function defined on some neighborhood $U$ of $\bar{x}$ such that

i) $V(\bar{x}) = 0$ and $V(x) > 0$ if $x \neq \bar{x}$.

ii) $\dot{V}(x) \leq 0$ in $U - \{\bar{x}\}$.

Discuss the stability of $\bar{x}$. (20 marks)

**QUESTION FIVE (20 MARKS)**

a) Consider the following vector field with time periodic coefficients;

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = A(t) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

Where

$$A(t) = \begin{pmatrix} -1 + \frac{3}{2}\cos^2(t) & 1 - \frac{3}{2}\cos(t)\sin(t) \\ -1 - \frac{3}{2}\cos(t)\sin(t) & -1 + \frac{3}{2}\sin^2(t) \end{pmatrix}.$$

i) Find the eigenvalues of $A(t)$. (4 marks)

ii) Find the solution(s) of the equation. (5 marks)

iii) Comment on the stability of the solution(s). (1 mark)

b) Suppose $f$ is a differential vector field with 0 as a hyperbolic fixed point. Denote $\Phi(t, x)$ the corresponding flow and $A = df_0$ the Jacobian matrix of $f$ at 0. Show that there is a homeomorphism $\varphi(x) = x + h(x)$ with $h$ bounded such that

$$\varphi \circ e^{tA} = \Phi_t \circ \varphi$$

in a sufficiently small neighborhood of 0. (10 marks)

**--END--**