UNIVERSITY OF EMBU

2017/2018 ACADEMIC YEAR

TRIMESTER EXAMINATIONS

SECOND YEAR EXAMINATION FOR THE DEGREE OF MASTER OF SCIENCE IN APPLIED MATHEMATICS

SMA 631: APPLIED DYNAMICAL SYSTEMS II

DATE: AUGUST 1, 2018 TIME: 2:00 PM – 5:00 PM

INSTRUCTIONS:

Answer Question ONE and ANY other two Questions. Exam Duration is 3 Hours.

QUESTION ONE (30 MARKS)

a) With the help of a diagram, discuss the bifurcation of the vector field, (5 marks)

$$\dot{x} = \mu x - x^2, \qquad x \in \mathbb{R}^1, \qquad \mu \in \mathbb{R}^1.$$

b) Let $H^1 \supset H^2 \supset \cdots \supset H^k \supset \cdots$ be a nested sequence of μ_h-horizontal strips with $d(H^k) \to 0$ as $k \to \infty$. Show that

$$\bigcap_{k=1}^{\infty} H^k$$

is a μ_h-horizontal curve. (5 marks)

c) Consider a $C^r (r \gg 1)$ autonomous one-dimensional vector field in x. Given $S \subset \mathbb{R}^n$ to be a compact set invariant under $\phi(t, x)$ for all $t \in \mathbb{R}$. When is S said to be chaotic? (6 marks)

d) Consider the one-parameter family of one-dimensional vector fields,

$$\dot{x} = f(x, \mu), \qquad x \in \mathbb{R}^1, \qquad \mu \in \mathbb{R}^1;$$

with $f(0,0) = 0$ and $\frac{\partial f}{\partial x}(0,0) = 0$.

If a one-parameter family of one-dimensional vector fields satisfying

$$f(0,0) = 0, \quad \frac{\partial f}{\partial x}(0,0) = 0, \quad \frac{\partial f}{\partial \mu}(0,0) \neq 0, \quad \text{and} \quad \frac{\partial^2 f}{\partial x^2}(0,0) \neq 0,$$

is perturbed, will the resulting family of one-dimensional vector fields have qualitatively the same dynamics? Explain. (6 marks)

e) Suppose we have $(\bar{t}_0, \bar{\mu})$ such that Melnikov's function M satisfies;

(i) $M(\bar{t}_0, \phi_0, \bar{\mu}) = 0$,

(ii) $\frac{\partial M}{\partial t_0}(\bar{t}_0, \phi_0, \bar{\mu}) = 0$,

(iii) $\frac{\partial M}{\partial \mu}(\bar{t}_0, \phi_0, \bar{\mu}) \neq 0$,

(iv) $\frac{\partial^2 M}{\partial t_0^2}(\bar{t}_0, \phi_0, \bar{\mu}) \neq 0$.

Show that the stable and unstable manifolds of the hyperbolic fixed point on the cross-section Σ^{ϕ_0} are quadratically tangent at $(q_0(-\bar{t}_0)) + O(\varepsilon)$ for $\mu = \bar{\mu} + O(\varepsilon)$.

(5 marks)

f) Very often, one hears the phrase, "A dynamical system is chaotic if it has one positive Lyapunov exponent". Is this statement true? Discuss this by use of an example.

(3 marks)

QUESTION TWO (20 MARKS)

Consider a vector field on $\mathbb{R}^n$ having a fixed point at which the matrix associated with the linearization of the vector field about the fixed point has two zero eigenvalues bounded away from the imaginary axis. The study of the dynamics near this nonhyperbolic fixed point can be reduced to the study of the dynamics of the vector field restricted to the associated center manifold. Discuss steps for the study of the dynamics near the nonhyperbolic fixed point, given the Jordan canonical form of the linear part to be $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. (20 marks)

QUESTION THREE (20 MARKS)

a) Consider the following vector field,

$$\dot{x} = y,$$
$$\dot{y} = x - x^2 - \varepsilon\delta y + \varepsilon\gamma \cos(\omega t).$$

Compute the Melnikov function and describe the surface in $\gamma - \delta - \omega$ space where the bifurcation to homoclinic orbits occurs. (12 marks)

b) Consider the vector field,

$$\begin{aligned} \dot{\theta} &= \varepsilon v, \\ \dot{v} &= -\varepsilon \sin(\theta) + \varepsilon^2 \gamma \cos(\omega t), \qquad (\theta, v) \in S^1 \times \mathbb{R}^1, \qquad \varepsilon \ll 1. \end{aligned}$$

Apply Melnikov's method to show that the Poincare map associated with this equation has transverse homoclinic orbits. (8 marks)

QUESTION FOUR (20 MARKS)

a) Consider the "Hopf-steady state interaction", i.e. a three-dimensional vector field having a fixed point where the (real) Jordan canonical form of the matrix is given by

$$\begin{pmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \omega > 0.$$

Show that this is a co-dimension two bifurcation and compute a candidate for a versal deformation. (8 marks)

b) Consider the following ODE:

$$\begin{aligned} \dot{x} &= \frac{\omega}{\sqrt{3}}(y - z) + [\varepsilon - \mu(x^2 - yz)]x, \\ \dot{y} &= \frac{\omega}{\sqrt{3}}(z - x) + [\varepsilon - \mu(y^2 - xz)]y, \quad (x, y, z) \in \mathbb{R}^3 \\ \dot{z} &= \frac{\omega}{\sqrt{3}}(x - y) + [\varepsilon - \mu(z^2 - xy)]z, \end{aligned}$$

where $\varepsilon > 0$, $\mu > 0$, and ω are parameters.

i) Show that for $\varepsilon = 0$, the eigenvalues of the matrix associated with the linearized vector field are given by $0, \pm\omega$. (6 marks)

ii) Study the bifurcations associated with these fixed points for ($\varepsilon = 0, \omega \neq 0$), and ($\varepsilon = 0, \omega = 0$). (6 marks)

QUESTION FIVE (20 MARKS)

a) Consider Lipschitz constants μ_v and μ_h such that $0 \leq \mu_v \mu_h < 1$. Show that a μ_v-vertical curve and a μ_h-horizontal curve intersect in a unique point. (10 marks)

b) Consider a linear velocity field given by

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} \cos(4t) & \sin(4t) - 2 \\ \sin(4t) + 2 & -\cos(4t) \end{pmatrix}.$$

i) Verify that the solution of the vector field is given by (5 marks)

$$x_1(t) = x_{10}e^{t}\cos(2t) - x_{20}e^{-t}\sin(2t)$$
$$x_2(t) = x_{10}e^{t}\sin(2t) + x_{20}e^{-t}\cos(2t).$$

ii) Compute the Lyapunov exponents associated with this fundamental solution matrix. (5 marks)

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