W1-2-60-1-6

# JOMO KENYATTA UNIVERSITY OF AGRICULTURE AND TECHNOLOGY

## UNIVERSITY EXAMINATIONS 2018/2019

## THIRD YEAR EXAMINATION FOR BACHELOR OF SCIENCE IN ACTUARIAL SCIENCE/BIOSTATISTICS/FINANCIAL ENGINEERING/STATISTICS/OPERATIONS RESEARCH

## SMA 2306: LINEAR ALGEBRA II

**DATE: DECEMBER 2018** **TIME: 2 HOURS**

**INSTRUCTIONS: Answer question ONE and any other TWO questions**

**QUESTION ONE**

a) Let $M_{nn}$ be the vector space of $n \times n$ matrices. Determine whether $T: M_{nn} \to M_{nn}$ defined by $T(A) = \det(A)$ is a linear transformation (4 marks)

b) Let $T: V \longrightarrow W$ be a linear mapping with $\dim V = n$ and $\dim W = m$. Show that

(i) If $\ker T = 0$ then $T$ is one-to-one (3 marks)

(ii) $T$ is onto if $\text{rank}(T) = m$. (2 marks)

c) In each part, use the given information to find the rank of the linear transformation $T$

i) $T: R^7 \to M_{32}$ has nullity 2 (1 mark)

ii) $T: P_3 \to \mathbb{R}$ has nullity 1 (1 mark)

iii) The null space of $T: P_5 \to P_5$ is $P_5$ (2 marks)

d) Let $f: R^3 \to R^3$ be defined by $f(x, y, z) = (2x - 3y, y - 3z, x + 4y)$ and let $\{(1, 1, 2), (0, 1, 1), (-1, 0, 1)\}$ be a basis for $R^3$. Find matrix $A$ of $f$ and a matrix $B$ which is similar to $A$. (5 marks)

e) Calculate $\begin{vmatrix} 4 & 0 & 0 & 1 & 0 \\ 3 & 3 & 3 & -1 & 0 \\ 1 & 2 & 4 & 2 & 3 \\ 9 & 4 & 6 & 2 & 3 \\ 2 & 2 & 4 & 2 & 3 \end{vmatrix}$ (5 marks)

f) Find a $3 \times 3$ matrix $B$ that has the eigenvalues 1, -1, and 0, and for which $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ are their corresponding eigenvectors (7 marks)

## QUESTION TWO

a) Let $A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & -1 & 0 & 3 \end{bmatrix}$

i. Calculate the eigenvalues and the bases of the Eigenspaces of $A$ (7 marks)
ii. Calculate the eigenvalues and the bases of the Eigenspaces of $A^8$ (3 marks)
iii. Find a matrix $P$ that diagonalizes $A$ and a matrix $D$ such that $D = P^{-1}AP$ (2 marks)
iv. Calculate the minimal polynomial of $A$ (4 marks)

b) If $B = \begin{bmatrix} \lambda - 4 & 0 & 0 \\ 0 & \lambda & 2 \\ 0 & 3 & \lambda - 1 \end{bmatrix}$, find all values of $\lambda$ for which $\det B = 0$ (4 marks)

## QUESTION THREE

a) Let $T: P_3 \to P_2$ be the mapping defined by $T(a_0 + a_1x + a_2x^2 + a_3x^3) = 5a_0 + a_3x^2$
i) Find a basis for the kernel of $T$ (2 marks)
ii) Find a basis for the range of $T$ (2 marks)

b) If $T: V \to W$ is a linear transformation, show that the kernel of $T$ is a subspace of $V$ (3 marks)

c) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Show that
i) Show that $A$ is diagonalizable if $(a-d)^2 + 4bc > 0$ (3 marks)
ii) Show that $A$ is not diagonalizable if $(a-d)^2 + 4bc < 0$ (2 marks)

d) Consider the basis $S = \{v_1, v_2\}$ for $R^2$, where $v_1 = (-2, 1)$ and $v_2 = (1, 3)$, and let $T: R^2 \to R^3$ be the linear transformation such that $T(v_1) = (-1, 2, 0)$ and $T(v_2) = (0, -3, 5)$. Find a formula for $T(x_1, x_2)$ and use the formula to find $T(2, -3)$ (5 marks)

e) Show that the matrices $B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ are similar (3 marks)

## QUESTION FOUR

a) It can be proved that if a square matrix $M$ is partitioned into ***block triangular form*** as $M = \begin{bmatrix} A & 0 \\ C & B \end{bmatrix}$ or $M = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$, in which $A$ and $B$ are square matrices, then $\det(M) = \det(A)\det(B)$.
Use this result to compute the determinant of the matrix

$$M = \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 8 & 6 & -9 \\ 2 & 5 & 0 & 4 & 7 & 5 \\ -1 & 3 & 2 & 6 & 9 & -2 \\ \hline 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & -3 & 8 & -4 \end{array}\right]$$

(3 marks)

b) Show that the matrix $A = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & -\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is invertible for all values of $\theta$; then find $A^{-1}$

(7 marks)

c) Let $B = \begin{bmatrix} -1 & 7 & -1 \\ 0 & 1 & 0 \\ 0 & 15 & -2 \end{bmatrix}$ and $P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 5 \end{bmatrix}$

i) Confirm that $P$ diagonalizes $B$ (6 marks)

ii) Determine the eigenvalues of $B$ (1 mark)

iii) Compute $B^{11}$ (3 marks)